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Statistics of Nonlinearly Transformed Coherence Estimates

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Preface

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An asymptotic relation for the average value of any nonlinear transformation of a magnitude-squared-coherence estimate is derived. It is utilized to determine the bias, variance, and mean-square error of the transformed coherence estimate. It is found that the arc tanh (x^2) transformation is the only one with variance independent of the true coherence value to $O(N^{-1})$, and that the ln(x) transformation is the only one with zero bias, to exponential order in N. Here, N is the number of data pieces employed in the estimation of the coherence. Extensions to terms of $O(N^{-2})$ are made and presented for a ν-th law device also.

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List of Symbols

$O(x)$	order of x
N	number of data pieces used in coherence estimate
C	true value of magnitude-squared coherence
$\hat{C}$	estimate of C
MSC	magnitude-squared coherence
MC	magnitude coherence
$g(\hat{C})$, $g(x)$	output of nonlinear transformation of $\hat{C}$, x
$p(x)$	probability density function of $\hat{C}$
E	ensemble average
A	average output of nonlinear transformation
ν_n	n-th moment of $\hat{C}$ about C
B	bias at output of nonlinear transformation
Q	mean-square output of nonlinear transformation
V	variance of output of nonlinear transformation
S	mean-square error at output of nonlinear transformation

STATISTICS OF NONLINEARLY TRANSFORMED COHERENCE ESTIMATES

INTRODUCTION

Previous results on the statistics of the magnitude-squared-coherence (MSC) estimate $\hat{C}$ and the magnitude-coherence (MC) estimate $\hat{C}^{1/2}$ have required simplification of hypergeometric functions $_3F_2$, sometimes with great labor (references 1-4). This effort has been required because of the paucity of asymptotic results for the $_3F_2$ function. In reference 4, this shortcoming was partially alleviated by deriving asymptotic results for the MSC estimate, and doing curve-fitting for the MC estimate, thereby obtaining relatively simple relations for the bias, variance, and mean-square error in these two cases.

However, whenever a different nonlinear transformation of the MSC estimate is considered, the analytical effort must begin anew to determine the fundamental behavior of the statistics such as moments. For example, in reference 5, the nonlinear transformation arc tanh $(\hat{C}^{1/2})$ of MSC estimate $\hat{C}$ was shown to yield a nearly-Gaussian random variable, thereby faciiitating calculation of confidence limits for coherence detectors. However, the mean and variance of the nearly Gaussian random variable were deduced by a time-consuming trial-and-error curve-fitting procedure.

Here, we will rectify this situation by deriving simple asymptotic relations, for large N, for the statistics of any nonlinear transformation of $\hat{C}$, where N is the number of data pieces employed in the estimation of coherence (reference 6). In this fashion, we can determine the fundamental behavior of statistics like the bias, variance, and mean-square error for a particular distortion of $\hat{C}$, without an undue amount of labor. Also, we can deduce new nonlinear transformations with desirable behavior.

ESTIMATION OF MAGNITUDE-SQUARED COHERENCE

The complex coherence between two jointly-stationary random processes x(t) and y(t) is defined as

$$\gamma_{xy}(f) = \frac{G_{xy}(f)}{\left[G_{xx}(f)\ G_{yy}(f)\right]^{1/2}}, \tag{1}$$

where $G_{xy}(f)$ is the cross-spectral density at frequency f, and $G_{xx}(f)$ and $G_{yy}(f)$ are the auto-spectral densities. The MSC is

$$C(f) = |\gamma_{xy}(f)|^2 . \tag{2}$$

The MSC is frequently estimated according to (reference 6)

$$\hat{C}(f) \equiv \frac{|\hat{G}_{xy}(f)|^2}{\hat{G}_{xx}(f)\ \hat{G}_{yy}(f)} = \frac{\left|\sum_{n=1}^{N} X_n(f)\ Y_n^*(f)\right|^2}{\sum_{n=1}^{N} |X_n(f)|^2\ \sum_{n=1}^{N} |Y_n(f)|^2}, \tag{3}$$

where N is the number of data segments employed, and $X_n(f)$, $Y_n(f)$ are the (discrete) Fourier transforms of the n-th weighted data segments of x(t) and y(t).

The statistics of a nonlinearly transformed version $g(\hat{C})$ of MSC estimate $\hat{C}$ are of interest here. We drop frequency dependence f henceforth, for notational simplicity; thus C is the true (unknown) value of MSC that we are estimating.

AVERAGE VALUE OF TRANSFORMED COHERENCE ESTIMATE

The probability density function of MSC estimate $\hat{C}$ is given by reference 1, eq. (2) et seq., as

$$p(x) = (N-1)\left(\frac{1-C}{1-Cx}\right)^2 \left[\frac{(1-C)(1-x)}{1-Cx}\right]^{N-2} P_{N-1}\left(\frac{1+Cx}{1-Cx}\right)$$

$$\text{for } 0 \leq x \leq 1,\ 0 \leq C < 1 \quad , \tag{4}$$

where P_{N-1} is a Legendre polynomial. If $\hat{C}$ is subjected to nonlinear transformation $g(\hat{C})$, the average value of the output is

$$A = E\{g(\hat{C})\} = \int_0^1 dx\ p(x)\ g(x) \ . \tag{5}$$

For large N, probability density p(x) is peaked about* $x = C$; see reference 1, figures 1a-1h. Accordingly, the major contribution to (5) will come from this neighborhood, so we expand transformation g about this point:

$$A = \int_0^1 dx\ p(x) \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(C)\ (x-C)^n = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(C)\ \nu_n \ , \tag{6}$$

where

$$\nu_n \equiv \int_0^1 dx\ p(x)\ (x-C)^n \tag{7}$$

is the n-th moment of estimate $\hat{C}$ about true value C.

An expression for general moment $E\{\hat{C}^m\}$ is given in reference 4, page 2, along with specific simpler results, for m = 1 and 2, in terms of the Gauss hypergeometric function ${}_2F_1$. In appendix A here, ν_1 and ν_2 are developed in an asymptotic expansion through order N^{-2}; in appendix B, the dominant behavior of ν_n is developed for all integer n. The results are

$$\nu_1 = \frac{(1-C)^2}{N} + \frac{2C(1-C)^2}{N^2} + O(N^{-3}) \quad ; \tag{8}$$

$$\nu_2 = \frac{2C(1-C)^2}{N} + \frac{2(1-C)^2\ (1-6C+7C^2)}{N^2} + O(N^{-3}) \quad ; \tag{9}$$

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*For $C = 1$, $p(x) = \delta(x-1)$, whereas for $C = 0$, $p(x) = (N-1)(1-x)^{N-2}$. We exclude these atypical cases from consideration here. These probability density functions for $C = 1$ and $C = 0$ are sufficiently simple that they can be investigated separately.

$$\nu_n \sim \left\{ \begin{array}{ll} \dfrac{n!\, C^{n/2}(1 - C)^n}{(n/2)!\, N^{n/2}} & \text{for n even} \\ \\ \dfrac{n!\, C^{\frac{n-1}{2}}\,(1 - C)^n \left[1 + C + n(1 - 3C)\right]}{2\left(\frac{n-1}{2}\right)!\, N^{\frac{n+1}{2}}} & \text{for n odd} \end{array} \right\} \begin{array}{l} \text{as } N \to \infty \,; \\ C > 0 \quad . \end{array} \tag{10}$$

In particular, it is seen that ν_1 and ν_2 are $O(N^{-1})$, ν_3 and ν_4 are $O(N^{-2})$, and ν_n for $n \geq 5$ is $O(N^{-3})$ or smaller.

Combining these results in (6), we obtain for the average value of the device output:

$$A = g(C) + \frac{(1 - C)^2}{N}\left[g'(C) + Cg''(C)\right]$$
$$+ \frac{(1 - C)^2}{N^2}\Big[2Cg'(C) + (1 - 6C + 7C^2)\, g''(C)$$
$$+ 2C(1 - C)(1 - 2C)\, g'''(C) + \frac{1}{2} C^2 (1 - C)^2\, g''''(C)\Big] + O(N^{-3}). \tag{11}$$

This result gives the fundamental dependence on N, C, and transformation g. Through $O(N^{-1})$, we need to evaluate the nonlinear function and its first two derivatives, but to be correct to $O(N^{-2})$, it is necessary to evaluate up through the fourth derivative of the nonlinear function.

The bias at the output of the nonlinear device is defined as

$$B = E\{g(\hat{C})\} - g(C) = A - g(C) \quad , \tag{12}$$

and is given by the terms after g(C) in (11). It is generally of $O(N^{-1})$; this will be elaborated on later.

A particular example of the application of (11) is afforded by transformation $g(x) = x$. Then the device output is just the MSC estimate, and (12) yields bias

$$B = \frac{(1 - C)^2}{N} + \frac{2C(1 - C)^2}{N^2} + O(N^{-3}) \quad , \tag{13}$$

in agreement with reference 4, eq. (8).

ALTERNATIVE FORMS FOR AVERAGE VALUE

First-Order Approximations

Let us retain only the terms through $O(N^{-1})$ in (11) and (12); then

$$A \sim g(C) + \frac{(1 - C)^2}{N}\left[g'(C) + Cg''(C)\right] \quad \text{as } N \to \infty \quad , \tag{14}$$

and

$$B \sim \frac{(1 - C)^2}{N}\left[g'(C) + Cg''(C)\right] \quad \text{as } N \to \infty \quad . \tag{15}$$

Now we approximate the right-hand side of (14) by the quantity

$$A_1 \equiv g\left(C + \frac{a(C)}{N}\right) \quad , \tag{16}$$

which can be developed as

$$A_1 \sim g(C) + \frac{a(C)}{N}\, g'(C) \quad \text{as } N \to \infty \quad , \tag{17}$$

and we choose a(C) so that (14) and (17) match as N becomes large. There follows the approximation to the average,

$$A_1 = g\left(C + \frac{(1 - C)^2}{N}\left[1 + C\,\frac{g''(C)}{g'(C)}\right]\right) \quad . \tag{18}$$

The bias approximation follows from (12) and (18) as

$$B_1 \equiv g\left(C + \frac{(1 - C)^2}{N}\left[1 + C\,\frac{g''(C)}{g'(C)}\right]\right) - g(C) \quad . \tag{19}$$

Forms (18) and (19) turn out to be more useful than (14) and (15), respectively, in some cases. For the MSC estimate itself, $g(x) = x$, and the results yielded by (18) and (19) are identical to (14) and (15), respectively.

For the MC estimate $\hat{C}^{1/2}$, we have

$$g(x) = x^{1/2}, \quad g'(x) = \frac{1}{2}x^{-1/2}, \quad g''(x) = -\frac{1}{4}x^{-3/2} \quad . \tag{20}$$

Then the equations above yield

$$A \sim C^{1/2} + \frac{(1-C)^2}{4NC^{1/2}} \quad \text{as } N \to \infty \quad , \tag{21}$$

$$A_1 = \left(C + \frac{(1-C)^2}{2N}\right)^{1/2} \quad , \tag{22}$$

$$B \sim \frac{(1-C)^2}{4NC^{1/2}} \quad \text{as } N \to \infty \quad , \tag{23}$$

$$B_1 = \left(C + \frac{(1-C)^2}{2N}\right)^{1/2} - C^{1/2} \quad . \tag{24}$$

Whereas (21) and (23) blow up as $C \to 0$, (22) and (24) remain finite. In fact, (24) is precisely the result obtained by curve-fitting in reference 4, page (20), eqs. (39) and (33), to $O(N^{-1})$.

This is a fortuitous situation for approximations (22) and (24). In the derivations above, it was presumed that transformation $g(x)$ could be well approximated in terms of a few derivatives at $x = C$ and that $C > 0$. These conditions are obviously violated for MC device (20) as $C \to 0$; nevertheless, forms (22) and (24) are still reasonable in this limit. This situation will arise for other nonlinear transformations considered below. The basic problem is that (21) is not a uniform asymptotic expansion with respect to C (reference 7, chapter 9); it holds as $N \to \infty$, for fixed $C > 0$.

For the transformation yielding a nearly Gaussian random variable, discussed in the Introduction and studied in reference 5, we have

$$g(x) = \text{arc tanh}(x^{1/2}), \quad g'(x) = \frac{1}{2x^{1/2}(1-x)} \quad ,$$
$$g''(x) = \frac{3x-1}{4x^{3/2}(1-x)^2} \quad . \tag{25}$$

Application of (14), (15), (18), and (19) yields

$$A \sim \text{arc tanh}(C^{1/2}) + \frac{1+C}{4NC^{1/2}} \quad \text{as } N \to \infty \quad , \tag{26}$$

$$A_1 = \text{arc tanh}\left(\left(C + \frac{1 - C^2}{2N}\right)^{1/2}\right) , \tag{27}$$

$$B \sim \frac{1 + C}{4NC^{1/2}} \quad \text{as } N \to \infty , \tag{28}$$

$$B_1 = \text{arc tanh}\left(C + \frac{1 - C^2}{2N}\right) - \text{arc tanh}(C) . \tag{29}$$

Result (27) agrees with reference 5, eqs. (7) and (8), which was obtained only after considerable trial and error and curve-fitting. Again, the approximations (27) and (29) are better behaved than their progenitors (26) and (28), which blow up as $C \to 0$. As above, (26) is not a uniform asymptotic expansion with respect to C.

Finally, let us inquire into what nonlinear transformation would yield zero bias, to $O(N^{-1})$. Equation (15) indicates that the only such device must satisfy

$$\frac{d}{dC}\{Cg'(C)\} = 0 . \tag{30}$$

That is, the logarithmic transformation

$$g(x) = a \ln(x) + b \tag{31}$$

has zero bias, for any constants a and b, to $O(N^{-1})$.

Second-Order Approximations

We now retain terms through $O(N^{-2})$ in the average (11) and represent it by shorthand notation:

$$A \sim g(C) + \frac{\alpha(C)}{N} + \frac{\beta(C)}{N^2} \quad \text{as } N \to \infty . \tag{32}$$

We now want to fit this asymptotic expansion by the approximation

$$A_2 \equiv g\left(C + \frac{a(C)}{N} + \frac{b(C)}{N^2}\right) \tag{33}$$

to $O(N^{-2})$. Accordingly we develop (33) as

$$A_2 \sim g(C) + g'(C)\left(\frac{a(C)}{N} + \frac{b(C)}{N^2}\right) + \frac{1}{2} g''(C) \frac{a^2(C)}{N^2} \quad \text{as } N \to \infty . \tag{34}$$

Comparison of (32) and (34) yields

$$a(C) = \frac{\alpha(C)}{g'(C)} \quad , \quad b(C) = \frac{\beta(C)}{g'(C)} - \frac{g''(C)\ \alpha^2(C)}{2[g'(C)]^3} \quad , \tag{35}$$

where from (11), we already know

$$\alpha(C) = (1 - C)^2 \left[g'(C) + Cg''(C)\right] \ ,$$

$$\beta(C) = (1 - C)^2 \Big[2Cg'(C) + (1 - 6C + 7C^2)\ g''(C) + 2C(1 - C)\ (1 - 2C)\ g'''(C) + \frac{1}{2}\,C^2(1 - C)^2\ g''''(C)\Big] \ . \tag{36}$$

Equation (33), together with (35) and (36), gives the second-order approximation to the average value. The result for a(C) in (35) is identical to that obtained earlier in (18); that for b(C) is quite involved and requires a specific transformation for simplification.

The results for bias are now obtained by combining (12) and (32),

$$B \sim \frac{\alpha(C)}{N} + \frac{\beta(C)}{N^2} \quad \text{as } N \to \infty \quad , \tag{37}$$

and by combining (12) and (33),

$$B_2 \equiv g\left(C + \frac{a(C)}{N} + \frac{b(C)}{N^2}\right) - g(C) \quad . \tag{38}$$

Thus our general results to $O(N^{-2})$ are given by (32), (33), (37), and (38), where the various parameters are given in (35) and (36). Now we apply these results to specific examples.

For the MSC estimate itself,

$$g(x) = x \ , \ g'(x) = 1, \ g''(x) = g'''(x) = g''''(x) = 0 \quad , \tag{39}$$

and there follows

$$a = \alpha = (1 - C)^2, \ b = \beta = 2C(1 - C)^2 \ ,$$

$$A \sim A_2 = C + \frac{(1 - C)^2}{N} + \frac{2C(1 - C)^2}{N^2} \quad , \tag{40}$$

$$B \sim B_2 = \frac{(1 - C)^2}{N} + \frac{2C(1 - C)^2}{N^2} \quad .$$

This last relation has already been noted in (13).

For the MC estimate, we have

$$g(x) = x^{1/2},\ g'(x) = \frac{1}{2}x^{-1/2},\ g''(x) = -\frac{1}{4}x^{-3/2},$$

$$g'''(x) = \frac{3}{8}x^{-5/2},\ g''''(x) = -\frac{15}{16}x^{-7/2}. \tag{41}$$

Equation (36) yields

$$\alpha = \frac{(1-C)^2}{4C^{1/2}},\quad \beta = \frac{(1-C)^2(1+3C)^2}{32C^{3/2}}, \tag{42}$$

and (35) yields

$$a = \frac{(1-C)^2}{2},\ b = \frac{(1-C)^2(1+2C+5C^2)}{8C}. \tag{43}$$

Then (32) and (33) yield, for the results on the average value for the MC estimate,

$$A \sim C^{1/2} + \frac{(1-C)^2}{4NC^{1/2}} + \frac{(1-C)^2(1+3C)^2}{32N^2C^{3/2}} \quad \text{as } N \to \infty, \tag{44}$$

$$A_2 = \left(C + \frac{(1-C)^2}{2N} + \frac{(1-C)^2(1+2C+5C^2)}{8N^2C}\right)^{1/2}, \tag{45}$$

while (37) and (38) give the corresponding results on bias; namely, subtract C from each of the expressions in (44) and (45).

We now observe the disconcerting result that inclusion of the $O(N^{-2})$ term in approximation (45) blows up at $C=0$, whereas the $O(N^{-1})$ term does not. This suggests that, for small C, whereas approximation A_1 in (16) was well-suited to the available information to $O(N^{-1})$, there is a more suitable approximation than A_2 in (33) when $O(N^{-2})$ information is available on the average value; this possibility is taken up in appendix C. Basically, the fact that (44) is not a uniform asymptotic expansion, with respect to C, is causing this singularity at $C=0$. However, (44) and (45) yield very good approximations for large N and $C>0$.

For the arc tanh $(x^{1/2})$ transformation, we augment (25) with the two additional terms

$$g'''(x) = \frac{3 - 10x + 15x^2}{8x^{5/2}(1-x)^3},\ g''''(x) = \frac{3(35x^3 - 35x^2 + 21x - 5)}{16x^{7/2}(1-x)^4}. \tag{46}$$

Then from (36),

$$\alpha = \frac{1 + C}{4C^{1/2}} , \ \beta = \frac{(1 + C)\ (1 + 6C + C^2)}{32C^{3/2}} , \tag{47}$$

and from (35)

$$a = \frac{1 - C^2}{2} , \ b = \frac{(1 - C^2)\ (1 + 2C - C^2)}{8C} . \tag{48}$$

The average output can now be obtained from (32) and (33) in the forms

$$A \sim \text{arc tanh}\ (C^{1/2}) + \frac{1 + C}{4NC^{1/2}} + \frac{(1 + C)\ (1 + 6C + C^2)}{32N^2C^{3/2}} \text{ as } N \to \infty , \tag{49}$$

$$A_2 = \text{arc tanh}\left(\left(C + \frac{1 - C^2}{2N} + \frac{(1 - C^2)\ (1 + 2C - C^2)}{8N^2C}\right)^{1/2}\right) . \tag{50}$$

Since arc tanh $(x^{1/2}) \sim x^{1/2}$ as $x \to 0$, the same behavior regarding the blowup at $C = 0$ of the $O(N^{-2})$ term is expected and present, just as for the MC estimate relations in (44) and (45). The bias of the arc tanh $(x^{1/2})$ transformation is obtained by subtracting arc tanh $(C^{1/2})$ from both forms in (49) and (50). A modification of approximation (50), better suited to small C, is developed in appendix C.

For the logarithmic transformation of the MSC estimate, we have

$$g(x) = \ln(x),\ g'(x) = x^{-1},\ g''(x) = -x^{-2},$$
$$g'''(x) = 2x^{-3},\ g''''(x) = -6x^{-4} . \tag{51}$$

Equation (36) yields the surprising result

$$\alpha = 0 , \quad \beta = 0 , \tag{52}$$

for which (35) immediately gives

$$a = 0 , \quad b = 0 . \tag{53}$$

Thus both (32) and (33) yield for the average value of the device output

$$A \sim A_2 = \ln(C) + O(N^{-3}) . \tag{54}$$

Thus the unbiased character of the logarithmic distortion holds to at least $O(N^{-2})$, not just $O(N^{-1})$ as claimed originally in (30) and (31). This behavior in (54) holds of course for $C>0$, as noted earlier.

In appendix D, the average value of the ln(x) transformation is derived exactly; it is given by

$$A = -\sum_{n=1}^{N-1} \frac{(1 - C)^n}{n} \quad \text{for all C and N.} \tag{55}$$

By completing the summation to infinity, and then subtracting this added quantity, we can write (55) as

$$A = \ln(C) + \sum_{n=N}^{\infty} \frac{(1 - C)^n}{n} \quad \text{for } C > 0 \quad , \tag{56}$$

which has no approximations whatsoever. The summation in (56) is the bias and can be upper-bounded by

$$B < \frac{(1 - C)^N}{NC} = \frac{\exp(N \ln(1 - C))}{NC} \quad \text{for } C > 0 \quad . \tag{57}$$

Thus the decay of the bias is exponential in N for fixed $C>0$. This explains why the coefficients of N^{-1} and N^{-2} were zero in (53); in fact, all coefficients of N^{-k} would be zero for $k \geq 1$.

We should also observe (reference 8, eqs. 6.3.2 and 6.3.18) that average

$$A \sim \left\{ \begin{array}{ll} \ln(C) + \dfrac{(1 - C)^N}{NC} & \text{for } C > 0 \\ -\gamma - \ln(N) + \dfrac{1}{2N} + \dfrac{1}{2N^2} & \text{for } C = 0 \end{array} \right\} \quad \text{as } N \to \infty \quad . \tag{58}$$

Thus as the number of pieces, N, used in the MSC estimate increases, the average A saturates at ln(C) for $C \neq 0$, but gets arbitrarily negative for $C = 0$. This is due to the singularity of the transformation ln(x) at $x = 0$.

The last nonlinear transformation we consider is the ν-th law device:

$$g(x) = x^{\nu}, \; g^{(n)}(x) = (\nu + 1 - n)_n \, x^{\nu - n} \quad . \tag{59A}$$

Substitution in (35) and (36) yields, after simplification,

$$\alpha = \nu^2 (1 - C)^2 \, C^{\nu - 1} \quad ,$$

$$\beta = \frac{1}{2} \nu^2 (1 - C)^2 \, C^{\nu - 2} \, [1 - \nu + (1 + \nu) \, C]^2 \quad , \tag{59B}$$

and

$$a = \nu(1 - C)^2 \quad ,$$

$$b = \frac{\nu(1 - C)^2}{2C} \left[(1 + C)^2 - \nu(1 - C)\,(1 + 3C) \right] \quad , \tag{60}$$

respectively. The average output of the ν-th law device is then available upon substitution of (60) in (32) or (33). These results reduce to (40) for $\nu = 1$, and to (42) through (45) for $\nu = 1/2$. It can be seen from (60) that the only case where b does not tend to infinity, as C tends to zero, is for $\nu = 1$, the MSC estimate. All other cases do not yield a uniform asymptotic expansion in powers of N^{-1}, with respect to C. A modification of (60) to circumvent the singularity at $C = 0$ is presented in appendix C.

VARIANCE OF TRANSFORMED COHERENCE ESTIMATE

In order to determine the variance of the transformed coherence estimate, we need to be able to evaluate, in addition to (5), the average

$$Q \equiv E\{g^2(\hat{C})\} = \int_0^1 dx\ p(x)\ g^2(x) = \int_0^1 dx\ p(x)\ q(x) \quad , \tag{61}$$

where we have defined

$$q(x) = g^2(x) \quad . \tag{62}$$

But now we can use average output (11) on (61), with a re-identification of g in (11) as q here. So we need the quantities (using an obvious shorthand notation)

$$q = g^2,\ q' = 2gg'\ ,\ q'' = 2gg'' + 2g'g'\ ,\ q''' = 2gg''' + 6g'g'' \quad ,$$
$$q'''' = 2gg'''' + 8g'g''' + 6g''g'' \quad , \tag{63}$$

where all these functions are evaluated at C. There follows from (11),

$$\begin{aligned} Q = g^2 + \frac{2(1-C)^2}{N}[gg' + Cgg'' + Cg'g'] + \frac{(1-C)^2}{N^2}[4Cgg' \\ + 2(1-6C+7C^2)(gg''+g'g') \\ + 4C(1-C)(1-2C)(gg'''+3g'g'') \\ + C^2(1-C)^2(gg''''+4g'g'''+3g''g'')] + O(N^{-3}) \quad . \end{aligned} \tag{64}$$

The variance of the device output is then

$$V = Q - A^2 \ . \tag{65}$$

Substitution of (11) (as is) and (64) in (65) and cancellation of a number of similar terms yield the desired result for the variance of the device output:

$$\begin{aligned} V = \frac{2C(1-C)^2}{N} g'g' + \frac{(1-C)^2}{N^2}[g'g'(1-10C+13C^2) \\ + g'g''2C(1-C)(5-11C) + g''g''2C^2(1-C)^2 \\ + g'g'''\ 4C^2(1-C)^2] + O(N^{-3}) \quad . \end{aligned} \tag{66}$$

Here $g' = g'(C)$, $g'' = g''(C)$, $g''' = g'''(C)$. We observe, that to $O(N^{-1})$, only the first derivative, $g'(C)$, is required for the variance; recall that to $O(N^{-1})$, the average, (11), required $g''(C)$ in addition. Also $g''''(C)$ does not enter (66), at least through $O(N^{-2})$.

The first example we apply (66) to is the MSC estimate, $g(x) = x$. There follows immediately

$$V = \frac{2C(1 - C)^2}{N} + \frac{(1 - C)^2 (1 - 10C + 13C^2)}{N^2} + O(N^{-3}) \ . \tag{67}$$

This result agrees with reference 4, eq. 9, to $O(N^{-2})$ when the latter is expanded in powers of N^{-1}.

Next consider the MC estimate as given in (41). Employment in (66) results in

$$V = \frac{(1 - C)^2}{2N} - \frac{(1 - C)^2}{8N^2 C} [1 + 2C - 11C^2] + O(N^{-3}) \ . \tag{68}$$

The first term in (68) agrees with that in reference 4, eq. (33), to $O(N^{-1})$. The second terms do not agree, due mainly to the C^{-1} dependence in (68); that is, once again, (68) is not a uniform asymptotic expansion with respect to C. A modification to the singular contribution is considered in appendix E; the result is given by (E-2):

$$\tilde{V} = \frac{(1 - C)^2}{2N} - \frac{(1 - C)^2 (1 - 3C)}{2N^2} \ . \tag{69}$$

This modification for the variance of the MC estimate agrees precisely with the terms through $O(N^{-2})$ of reference 4, eq. (33), when the latter is developed in a power series in N^{-1}.

The pertinent equations for the arc tanh $(x^{1/2})$ transformation are presented in (25) and (46). Their use in (66) for the variance yields

$$V = \frac{1}{2N} - \frac{1 - 6C + C^2}{8N^2 C} + O(N^{-3}) \ . \tag{70}$$

That is, to $O(N^{-1})$, the variance of the arc tanh $(x^{1/2})$ output is independent of the true value, C, of the MSC. This result has been noted and utilized before; see reference 9 and reference 5, eq. (9). Furthermore, from (66), it may be seen that the only device with variance independent of C, to $O(N^{-1})$, is in fact q_1 arc tanh $(x^{1/2})$ + q_2, where q_1 and q_2 are constants.

A modification to (70) is derived in (E-3) and (E-4), namely,

$$\tilde{V} = \frac{1}{2N} + \frac{1}{2N^2} \ . \tag{71}$$

That is, the modification indicates a variance independent of the true value C of the MSC, through $O(N^{-2})$.

For the logarithmic transformation, we substitute (51) in (66) and obtain

$$V = \frac{2(1 - C)^2}{NC} + \frac{(1 - C)^2 (1 + C)^2}{N^2 C^2} + O(N^{-3}) \quad . \tag{72}$$

Here, even the $O(N^{-1})$ term tends to infinity as $C \to 0$; this is due to the singularity of the $\ln(x)$ transformation at $x = 0$. Evaluation of the mean square value and variance of the device output is conducted in appendix D; in particular, for the anomalous situation at $C = 0$ in (72), we find the exact result (D-17):

$$V = \sum_{k=1}^{N-1} \frac{1}{k^2} \quad \text{for } C = 0 \quad . \tag{73}$$

Asymptotically this behaves as (D-18):

$$V = \frac{\pi^2}{6} - \frac{1}{N} - \frac{1}{2N^2} + O(N^{-3}) \quad \text{for } C = 0 \quad . \tag{74}$$

Thus, whereas the variance of the logarithmic device output tends to zero as $N \to \infty$ if $C > 0$, according to (72), the variance stabilizes at $\pi^2/6$ as $N \to \infty$ for $C = 0$. This means that increasing the number of pieces, N, employed in the MSC estimate will not help in reducing the fluctuations at the device output if $C = 0$. However, it should be recalled from (58) that the average device output becomes arbitrarily large negatively in the case of $C = 0$; thus the ratio of standard deviation to average value does decrease to zero as $N \to \infty$, for $C = 0$ as well as for $C > 0$. This particular behavior of the $\ln(x)$ transformation is due to the logarithmic singularity at $x = 0$.

For the ν-th law device, we employ (59) in (66) and find

$$\begin{aligned} V = {} & \frac{2\nu^2 (1 - C)^2 C^{2\nu - 1}}{N} \\ & + \frac{\nu^2 (1 - C)^2 C^{2\nu - 2}}{N^2} \left[(1 + C)^2 - 6\nu (1 + C)(1 - C) \right. \\ & \left. + 6\nu^2 (1 - C)^2 \right] + O(N^{-3}) \quad . \end{aligned} \tag{75}$$

For $\nu = 1$, this reduces to (67), while for $\nu = \frac{1}{2}$, it becomes (68). A modification to the singular component of (75) at $C = 0$ (for $\nu < 1$) is obtained in (E-6):

$$\tilde{V} = \frac{2\nu^2 (1 - C)^2 C^{2\nu - 1}}{N} \left[1 + \frac{2(1 - 3\nu(1 - C))}{N} \right] . \tag{76}$$

For $\nu = \frac{1}{2}$, this reduces to (69). The variance approximation (76) is not singular at $C = 0$, provided that $\nu \geq \frac{1}{2}$.

MEAN SQUARE ERROR OF TRANSFORMED COHERENCE ESTIMATE

The desired output of the nonlinear transformation is the non-random constant $g(C)$. However, the actual output is $g(\hat{C})$. We therefore form an average squared error as

$$S \equiv E\{[g(\hat{C}) - g(C)]^2\} . \tag{77}$$

But this can be expressed in terms of previously evaluated quantities according to

$$\begin{aligned} S &= E\{[g(\hat{C}) - E\{g(\hat{C})\} + E\{g(\hat{C})\} - g(C)]^2\} \\ &= E\{[g(\hat{C}) - E\{g(\hat{C})\}]^2\} + [E\{g(\hat{C})\} - g(C)]^2 = V + B^2 , \end{aligned} \tag{78}$$

where the cross-product term averages to zero. When we use the results of (11), (12), and (66), (78) yields the general result for the mean square error:

$$\begin{aligned} S = \frac{2C(1-C)^2}{N} g'g' + \frac{(1-C)^2}{N^2} [2g'g'(1 - 6C + 7C^2) \\ + 12g'g''C(1-C)(1-2C) + 3g''g''C^2(1-C)^2 \\ + 4g'g'''C^2(1-C)^2] + O(N^{-3}) . \end{aligned} \tag{79}$$

For the MSC estimate itself, $g(x) = x$, and (79) yields

$$S = \frac{2C(1-C)^2}{N} + \frac{2(1-C)^2 (1 - 6C + 7C^2)}{N^2} + O(N^{-3}) , \tag{80}$$

in agreement with reference 4, eq. (12), when the latter is expanded in powers of N^{-1}.

For the MC estimate, $g(x) = x^{1/2}$, and there follows

$$S = \frac{(1-C)^2}{2N} - \frac{(1-C)^2(1 + 6C - 23C^2)}{16N^2C} + O(N^{-3}) . \tag{81}$$

For the arc tanh $(x^{1/2})$ device, use of (25) and (46) in (79) leads to

$$S = \frac{1}{2N} - \frac{1 - 14C + C^2}{16N^2C} + O(N^{-3}) \quad . \tag{82}$$

To $O(N^{-1})$, the mean square error is independent of the true value, C, of the MSC.

For the logarithmic transformation, (79) gives

$$S = \frac{2(1 - C)^2}{NC} + \frac{(1 - C)^2 \, (1 + C)^2}{N^2 \, C^2} + O(N^{-3}) \quad . \tag{83}$$

This equals variance (72), of course, since the bias is zero to all orders N^{-k}, as shown in (57).

Finally, the ν-th law device $g(x) = x^\nu$ yields, with the help of (79),

$$S = \frac{2\nu^2(1 - C)^2 \, C^{2\nu-1}}{N} + \frac{\nu^2(1 - C)^2 \, C^{2\nu-2}}{N^2}$$

$$\bullet \, [(1 + C)^2 - 6\nu(1 + C)\,(1 - C) + 7\nu^2(1 - C)^2] + O(N^{-3}) \quad . \tag{84}$$

For $\nu = 1$, this specializes to (80), while for $\nu = \frac{1}{2}$, it becomes (81).

DISCUSSION AND SUMMARY

Asymptotic expressions for the average value, bias, variance, and mean square error of the output of a nonlinear transformation have been derived, to $O(N^{-2})$. Extensions to higher moments of the device output are easily accomplished by interpreting q in (61) as the appropriate moment of g, as was done in (62) for the second moment. For example, we could evaluate the third cumulant of $g(\hat{C})$ in this fashion.

The modifications adopted here to attempt to alleviate the singular behavior at $C = 0$ of the terms of $O(N^{-2})$, for the arc tanh $(x^{1/2})$ and ν-th law devices, are recognized to be incorrect. The reason for the seemingly anomalous behavior is that the results here are not uniform asymptotic expansions with respect to C. The proper way to handle these cases is to derive the appropriate uniform asymptotic expansions. However, this would likely be a time-consuming and tedious task; the methods of reference 7, chapter 9, would be very relevant in this regard.

It is now a simple matter to evaluate the statistics of any additional candidates for MSC estimate transformation, such as $-\ln(1-x)$, $-(1-x)^{\nu}$, $-\text{arc tanh}\,((1-x)^{1/2})$, for example. These devices are not necessarily suggested for actual use, but rather are obvious modifications of the ones considered here; they have been chosen to be monotonically increasing over (0, 1).

Appendix A

Asymptotic Development of First Two Moments

From (7), the n-th moment of $\hat{C}$ about C is

$$\nu_n = \int_0^1 dx \; p(x)(x - C)^n \quad . \tag{A-1}$$

Of course $\nu_0 = 1$, and $\nu_1 = \mu_1 - C$, where moment

$$\mu_n \equiv \int_0^1 dx \; p(x) \; x^n \quad . \tag{A-2}$$

We now employ reference 4, eq. (5) to obtain

$$\nu_1 = -C + \frac{1}{N} + \frac{N-1}{N+1} C \; F(1, 1; N+2; C) \quad . \tag{A-3}$$

At this point, we expand all the terms in (A-3) to $O(N^{-2})$; there follows

$$\nu_1 = -C + \frac{1}{N} + \left(1 - \frac{1}{N}\right)\left(1 - \frac{1}{N} + \frac{1}{N^2}\right) C \left(1 + \frac{C}{N+2} + \frac{2C^2}{(N+2)(N+3)}\right) + O(N^{-3})$$

$$= -C + \frac{1}{N} + \left(1 - \frac{2}{N} + \frac{2}{N^2}\right) C \left(1 + \frac{C}{N}\left(1 - \frac{2}{N}\right) + \frac{2C^2}{N^2}\right) + O(N^{-3}) \quad .$$

$$= \frac{(1 - C)^2}{N} + \frac{2C(1 - C)^2}{N^2} + O(N^{-3}) \tag{A-4}$$

after simplification. Also $\mu_1 = C + \nu_1$ gives μ_1 to $O(N^{-2})$.

For n = 2, (A-1) and (A-2) yield

$$\nu_2 = \mu_2 - 2C\mu_1 + C^2 \quad . \tag{A-5}$$

An expression for μ_2 is given in reference 4, eq. (6); we develop it in like fashion to (A-4):

$$\mu_2 = -N^2 + 2N - 2 + \frac{2}{N} + \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{N} + \frac{1}{N^2} - \frac{1}{N^3} + \frac{1}{N^4}\right)$$

$$\bullet (N^2 - NC + 2C) \; F + O(N^{-3}) \quad , \tag{A-6}$$

where

$$F = 1 + \frac{C}{N + 2} + \frac{2C^2}{(N + 2)(N + 3)}$$

$$+ \frac{6C^3}{(N + 2)(N + 3)(N + 4)} + \frac{24C^4}{N^4} + O(N^{-5}) \quad . \qquad \text{(A-7)}$$

We must keep terms to $O(N^{-4})$ to counter the N^2 terms in (A-6). Substitution of(A-7) in (A-6), expansion of all the products, and retention of the highest orders yields, after considerable manipulation,

$$\mu_2 = C^2 + \frac{4C(1 - C)^2}{N} + \frac{2(1 - C)^2(1 - 3C)^2}{N^2} + O(N^{-3}) \quad . \qquad \text{(A-8)}$$

Finally, substitution of (A-8) and (A-4) in (A-5) yields

$$\nu_2 = \frac{2C(1 - C)^2}{N} + \frac{2(1 - C)^2 \, (1 - 6C + 7C^2)}{N^2} + O(N^{-3}) \quad . \qquad \text{(A-9)}$$

Although continuation of the procedure above to higher moments such as μ_3 and μ_4 (and thence ν_3 and ν_4) is possible, it is extremely tedious and error prone. A useful asymptotic development of (A-1) is proferred in appendix B.

Appendix B

Asymptotic Development of n-th Moment

The n-th moment of interest is given by (7),

$$\nu_n = \int_0^1 dx \; p(x)(x - C)^n \quad , \tag{B-1}$$

while p(x) is given by (4). For large N, the probability density p(x) peaks in the neighborhood of $x = C$. To see this, we use the asymptotic expansion of the Legendre polynomial given in reference 10, page 194, eq. 8.21.1:

$$P_m(y) \sim \frac{\left(y + (y^2 - 1)^{1/2}\right)^{m+1/2}}{(2\pi m)^{1/2}(y^2 - 1)^{1/4}} \text{ as } m \to \infty \, , \quad \text{for } y > 1 \quad . \tag{B-2}$$

We now identify $m = N-1$ and $y = (1 + Cx)/(1-Cx)$; then (B-2) and (4) yield

$$p(x) \sim \frac{(N - 1)^{1/2} \; (1 - C)^N \; (1 - x)^{N-2}}{2\pi^{1/2} \; (Cx)^{1/4} \left(1 - \sqrt{Cx}\right)^{2N-1}} \quad \text{as } N \to \infty \, , \text{ for } Cx > 0 \; . \tag{B-3}$$

The function $(1-x)/(1- \sqrt{Cx})^2$ peaks at $x \approx C$; raising it to a power sharpens this peak. The exact location of this peak to $O(N^{-1})$ is considered in appendix F. Substitution of (B-3) in (B-1) yields

$$\nu_n \sim \frac{N^{1/2}(1 - C)^N}{2\pi^{1/2} \; C^{1/4}} \int dx(x - C)^n \, \frac{1 - \sqrt{Cx}}{x^{1/4}(1 - x)^2} \left[\frac{1 - x}{(1 - \sqrt{Cx})^2}\right]^N \quad \begin{array}{l} \text{as } N \to \infty \, , \\ \text{for } C > 0 \, , \end{array} \tag{B-4}$$

where the integration can be confined to the neighborhood of the peak of the bracketed function at $x = C$.

The general problem is now to determine the asymptotic behavior of the integral

$$I \equiv \int dx(x - C)^n \; f(x) \; [\phi(x)]^N \tag{B-5}$$

as $N \to \infty$, where the integration is confined to the neighborhood of the peak of $\phi(x)$ at $x = C$. This problem is undertaken later in this appendix; here we shall merely make the identifications and evaluate the required parameters. We use only the dominant term in that asymptotic expansion:

$$f(x) = \frac{1 - \sqrt{Cx}}{x^{1/4}(1 - x)^2} , \quad \phi(x) = \frac{1 - x}{\left(1 - \sqrt{Cx}\right)^2} ,$$

$$b_o = f_o = \frac{1}{C^{1/4}(1 - C)} , \quad b_1 = f_1 = \frac{7C - 1}{4C^{5/4}(1 - C)^2} ,$$

$$\phi_o = \frac{1}{1 - C} , \quad \phi_1 = 0 ,$$

$$\phi_2 = \frac{-1}{2C(1 - C)^3} , \quad \phi_3 = \frac{3(1 - 3C)}{4C^2(1 - C)^4} ,$$

$$\rho_2 = \frac{-1}{4C(1 - C)^2} , \quad \rho_3 = \frac{1 - 3C}{8C^2(1 - C)^3} ,$$

$$\beta_1 = \left(\frac{n}{2} + 1\right) \; 4C(1 - C)^2 \quad \text{for n odd.} \tag{B-6}$$

The result for integral I in (B-5) then becomes

$$I \sim \left\{ \begin{array}{ll} \dfrac{\Gamma\left(\frac{n+1}{2}\right) \left[4C(1 - C)^2\right]^{\frac{n+1}{2}}}{(1 - C)^N \, N^{(n+1)/2} \, C^{1/4}(1 - C)} & \text{for n even} \\[2ex] \dfrac{\Gamma\left(\frac{n}{2} + 1\right) \left[4C(1 - C)^2\right]^{\frac{n}{2}+1} \left[1 + C + n(1 - 3C)\right]}{(1 - C)^N \, N^{\frac{n}{2}+1} \, 4C^{5/4} \, (1 - C)^2} & \text{for n odd} \end{array} \right\} \quad \begin{array}{l} \text{as } N \to \infty , \\ \text{for } C > 0 . \end{array} \tag{B-7}$$

Substitution in (B-4) and simplification lead to

$$\nu_n \sim \left\{ \begin{array}{ll} \dfrac{n! \; C^{n/2} \; (1 - C)^n}{(n/2)! \; N^{n/2}} & \text{for n even} \\[2ex] \dfrac{n! \; C^{\frac{n-1}{2}} \; (1 - C)^n \left[1 + C + n(1 - 3C)\right]}{2\left(\frac{n-1}{2}\right)! \; N^{\frac{n+1}{2}}} & \text{for n odd} \end{array} \right\} \quad \begin{array}{l} \text{as } N \to \infty , \\ \text{for } C > 0 . \end{array} \tag{B-8}$$

This is the desired general result; particular values are

$$\nu_o \sim 1, \; \nu_1 \sim \frac{(1 - C)^2}{N}, \; \nu_2 \sim \frac{2C(1 - C)^2}{N},$$

$$\nu_3 \sim \frac{12C(1 - C)^3(1 - 2C)}{N^2}, \; \nu_4 \sim \frac{12C^2(1 - C)^4}{N^2},$$

$$\nu_5 \sim \frac{60C^2(1 - C)^5(3 - 7C)}{N^3}, \; \nu_6 \sim \frac{120C^3(1 - C)^6}{N^3},$$

$$\text{as } N \to \infty, \text{ for } C > 0. \tag{B-9}$$

Asymptotic Development of I

The integral of interest is given in (B-5), where $f(C) \neq 0$ and where n is an integer. $\phi(x)$ has a maximum at the point C which is interior to the range of integration. We shall use and extend the procedure given in reference 11, pages 272-274, to develop the asymptotic expansion. We also limit the derivation here to the case of n even; the extension to n odd is presented without derivation.

Let $\Delta = x - C$ and

$$\phi(x) \sim \phi_o + \frac{1}{2!}\phi_2\Delta^2 + \frac{1}{3!}\phi_3\Delta^3 + \frac{1}{4!}\phi_4\Delta^4 \quad \text{as } \Delta \to 0$$

$$= \phi_o\left[1 + \rho_2\Delta^2 + \rho_3\Delta^3 + \rho_4\Delta^4\right], \tag{B-10}$$

where

$$\phi_m \equiv \phi^{(m)}(C), \quad \rho_m \equiv \frac{\phi_m}{m!\,\phi_o}. \tag{B-11}$$

Also let

$$f(x) \sim f_o + f_1\Delta + \frac{1}{2!}f_2\Delta^2 \quad \text{as } \Delta \to 0$$

$$= b_o + b_1\Delta + b_2\Delta^2, \tag{B-12}$$

where

$$f_m \equiv f^{(m)}(C), \; b_m \equiv \frac{f_m}{m!}. \tag{B-13}$$

Substitution in (B-5) yields

$$I \sim \int d\Delta\, \Delta^n \left(b_o + b_1\, \Delta + b_2\, \Delta^2\right) \phi_o^N \exp\left(N\, \ln\left(1 + \rho_2\Delta^2 + \rho_3\Delta^3 + \rho_4\Delta^4\right)\right)$$

$$\sim \phi_o^N \int d\Delta\, \Delta^n \left(b_o + b_1\Delta + b_2\Delta^2\right) \exp\left(N\left(\rho_2\Delta^2 + \rho_3\Delta^3 + \rho_4\Delta^4 - \rho_2^2\Delta^4/2\right)\right)$$

$$= \phi_o^N \int d\Delta\, \Delta^n \left(b_o + b_1\Delta + b_2\Delta^2\right) \exp\left(-N|\rho_2|\Delta^2\right)$$

$$\cdot \exp\left(N\rho_3\Delta^3\right) \exp\left(N\left(\rho_4 - \rho_2^2/2\right)\Delta^4\right) , \qquad \text{(B-14)}$$

where we utilized the fact that $\rho_2 < 0$ since $\phi(x)$ peaks at $x = C$.

Now let $t = \Delta/N^{1/2}$ in (B-14) and define $q = (n+1)/2$; then

$$I \sim \frac{\phi_o^N}{N^q} \int dt\; t^n\left(b_o + \frac{b_1 t}{N^{1/2}} + \frac{b_2 t^2}{N}\right)$$

$$\cdot \exp\left(-|\rho_2|t^2\right) \exp\left(\frac{\rho_3 t^3}{N^{1/2}}\right) \exp\left(\frac{\rho_4 - \rho_2^2/2}{N}\, t^4\right)$$

$$\sim \frac{\phi_o^N}{N^q} \int dt\, \exp\left(-|\rho_2|t^2\right) t^n\left(b_o + \frac{b_1 t}{N^{1/2}} + \frac{b_2 t^2}{N}\right)$$

$$\cdot\left(1 + \frac{\rho_3 t^3}{N^{1/2}} + \frac{\rho_3^2 t^6}{2N}\right)\left(1 + \frac{\rho_4 - \rho_2^2/2}{N}\, t^4\right)$$

$$\sim \frac{\phi_o^N}{N^q} \int dt\, \exp\left(-|\rho_2|t^2\right) t^n\left(b_o + \frac{1}{N^{1/2}}\left(b_1 t + b_o\rho_3 t^3\right)\right.$$

$$\left. + \frac{1}{N}\left(b_o\left(\rho_4 - \rho_2^2/2\right) t^4 + b_o\rho_3^2 t^6/2 + b_1\rho_3 t^4 + b_2 t^2\right)\right)$$

$$\sim \frac{\phi_o^N\, \Gamma(q)}{(N|\rho_2|)^q}\left[b_o + \frac{1}{N}\left\{\beta_2 b_o\left(\rho_4 - \rho_2^2/2\right) + \beta_3 b_o \rho_3^2 + \beta_2 b_1 \rho_3 + \beta_1 b_2\right\}\right]$$

$$\text{as } N \to \infty , \text{ for } n \text{ even}, \qquad \text{(B-15)}$$

where

$$\beta_m \equiv \frac{(q)_m}{|\rho_2|^m\, (m-1)!} , \quad q = \frac{n+1}{2} \quad \text{for } n \text{ even} . \qquad \text{(B-16)}$$

The procedure for n odd is exactly as above, except that $\phi(x)$ and $f(x)$ must be expanded to one higher order than in (B-10) and (B-12). All other symbols are as defined above, except that now

$$q = \frac{n}{2} + 1 \quad \text{for } n \text{ odd} \ . \tag{B-17}$$

There follows for (B-5),

$$I \sim \frac{\phi_o^N \, \Gamma(q)}{\left(N|\rho_2|\right)^q} \left[b_1 + \beta_1 b_o \rho_3 + \frac{1}{N} \left\{ \beta_2 b_o \left(\rho_5 - \rho_3 \rho_2\right) \right.\right.$$

$$+ \left(2\beta_3 b_o \rho_3 + \beta_2 b_1\right)\left(\rho_4 - \rho_2^2/2\right)$$

$$\left.\left. + \beta_4 b_o \rho_3^3 + \beta_3 b_1 \rho_3^2 + \beta_2 b_2 \rho_3 + \beta_1 b_3 \right\} \right] \quad \text{as } N \to \infty \ , \ \text{for } n \text{ odd.} \tag{B-18}$$

The desired asymptotic expansion of integral (B-5) is given by (B-15) and (B-18) for large N, where N need not be an integer. In deriving (B-7) and (B-8) earlier, we only used the dominant or leading term of (B-15) and (B-18). Extension of (B-7) and (B-8) to the next term would require consideration of the correction terms in (B-15) and (B-18).

For $n = 0$, the leading term of (B-15) reduces to reference 7, pages 211-212, eq. 5.6.21. Thus we have generalized here to nonzero n, both even and odd, and to the first correction term.

Appendix C

Modified Approximations For Average Value

The approximation A_2 developed for the average of the MC estimate in (45) tends to infinity as $C \rightarrow 0$. That is, the b(C) term, (43), which is used in approximation (33), blows up at C = 0. In order to rectify this situation, we reconsider the coefficients

$$a = \frac{(1-C)^2}{2}, \quad b = \frac{(1-C)^2\left(1+2C+5C^2\right)}{8C} \quad . \qquad \text{(C-1)}$$

Both a and b have the common factor $(1-C)^2$; this was also true for the MSC results in (40). Keeping this factor, we rewrite b in (C-1) as

$$b = \frac{(1-C)^2\left[(1-C)^2 + 4C(1+C)\right]}{8C}$$

$$= \frac{(1-C)^4}{8C} + \frac{(1-C)^2(1+C)}{2} \quad . \qquad \text{(C-2)}$$

The lead term in (C-2) has a fourth-order zero at C = 1, and a smaller scale factor; therefore its neglect would not significantly affect the value of (C-2) for moderate C. Also this term contains all of the singularity at C = 0. Moreover, since (45) is an approximation to an asymptotic expansion which is not uniform with respect to C, a reasonable modification to (C-2) is to drop the singularity while trying to realize as little effect on larger C values as possible. Accordingly we adopt modified coefficient

$$\tilde{b} = \frac{(1-C)^2(1+C)}{2} , \qquad \text{(C-3)}$$

thereby realizing the modified approximation to (45) for the MC estimate,

$$\tilde{A}_2 = \left(C + \frac{(1-C)^2}{2N} + \frac{(1-C)^2(1+C)}{2N^2}\right)^{1/2} \quad . \qquad \text{(C-4)}$$

This result agrees precisely to $O(N^{-2})$ with reference 4, eqs. (39) and (33), when the latter are expanded in a power series in N^{-1}; observe that $D = O(N^{-2})$ there.

Notice that we have modified the approximation (45), but not the asymptotic expansion (44); we must accept the asymptotic expansion as it is, since it is the unique expansion in powers of N^{-1} for this problem. But we can do what we please with an approximation.

For the arc tanh $(x^{1/2})$ transformation, coefficients a and b are given by (48) as

$$a = \frac{1-C^2}{2}, \quad b = \frac{\left(1-C^2\right)\left(1+2C-C^2\right)}{8C} \quad . \qquad \text{(C-5)}$$

We regroup the terms in b (keeping the common factor $1-C^2$ present in both terms) so as to minimize the effect for larger C:

$$b = \frac{1 - C^2}{8C}\left[(1 - C)^2 + 2C(2 - C)\right]$$

$$= \frac{(1 + C)(1 - C)^3}{8C} + \frac{\left(1 - C^2\right)(2 - C)}{4} \quad . \tag{C-6}$$

The modified coefficient is obtained by dropping the singular term:

$$\tilde{b} = \frac{\left(1 - C^2\right)(2 - C)}{4} \quad . \tag{C-7}$$

The modified approximation for the average value is then

$$\tilde{A} = \text{arc tanh}\left(\left(C + \frac{1 - C^2}{2N} + \frac{\left(1 - C^2\right)(2 - C)}{4N^2}\right)^{1/2}\right) . \tag{C-8}$$

This is an extension of reference 5, eqs. (7) and (8).

For the ν-th law device considered in (58), the coefficients are repeated from (60):

$$a = \nu(1 - C)^2 \;, \; b = \frac{\nu(1 - C)^2}{2C}\left[(1 - \nu) + 2(1 - \nu)C + (1 + 3\nu)C^2\right] . \tag{C-9}$$

Again, regrouping the terms in b, and keeping the common factor $(1-C)^2$, we obtain

$$b = \frac{\nu(1 - C)^2}{2C}\left[(1 - \nu)(1 - C)^2 + 4C(1 - \nu + \nu C)\right]$$

$$= \frac{\nu(1 - \nu)(1 - C)^4}{2C} + 2\nu(1 - C)^2 (1 - \nu + \nu C) \quad , \tag{C-10}$$

where the singular factor has the fourth-power of (1-C). The modified coefficient follows as

$$\tilde{b} = 2\nu(1 - C)^2 (1 - \nu + \nu C) \quad . \tag{C-11}$$

(This reduces to (C-3) for $\nu = ½$. For $\nu = 1$, the lead term in (C-10) is already zero, and (C-10) equals (40).) The modified approximation for the average value of the output of the ν-th law device is

$$\tilde{A}_2 = \left(C + \frac{\nu(1 - C)^2}{N} + \frac{2\nu(1 - C)^2 (1 - \nu + \nu C)}{N^2}\right)^{\nu} \quad . \tag{C-12}$$

An approximation to the bias is afforded by subtracting C^{ν} from (C-12).

Appendix D

Statistics of Logarithmic Transformation

Average Value

According to (5), the average value of the output of the logarithmic transformation is given by

$$A = \int_0^1 dx\ p(x)\ \ln x \quad . \tag{D-1}$$

Instead of using (4) for the probability density, we use reference 1, eq. (3):

$$A = \int_0^1 dx\ (N - 1)(1 - C)^N\ (1 - x)^{N-2}\ F(N, N; 1; Cx)\ \ln x$$

$$= (N - 1)(1 - C)^N \sum_{k=0}^{\infty} \frac{(N)_k\ (N)_k\ C^k}{(1)_k\ (1)_k} \int_0^1 dx(1 - x)^{N-2}\ x^k\ \ln x \quad . \tag{D-2}$$

We now use reference 12, eq. 4.253 1, to get, after *simplification*,

$$A = -(1 - C)^N \sum_{k=0}^{\infty} \frac{(N)_k\ C^k}{k!} \left[\psi(k + N) - \psi(k + 1)\right] \quad . \tag{D-3}$$

Then we employ reference 8, eqs. 6.3.6, 15.1.1, and 15.3.3 to develop

$$A = -(1 - C)^N \sum_{k=0}^{\infty} \frac{(N)_k\ C^k}{k!} \sum_{j=0}^{N-2} \frac{1}{k + 1 + j}$$

$$= -(1 - C)^N \sum_{k=0}^{\infty} \frac{(N)_k\ C^k}{k!} \sum_{j=0}^{N-2} \int_0^1 dx\ x^{k+j} \quad , \tag{D-4}$$

with the latter step taken to realize a product of k and j dependence. Interchanging summation and integration, and performing the sums (reference 8, eq. 15.1.8), we obtain

$$A = -(1 - C)^N \int_0^1 dx\ (1 - Cx)^{-N}\ \frac{1 - x^{N-1}}{1 - x} \quad . \tag{D-5}$$

Now let

$$y = \frac{x}{1 - x}\ , \quad x = \frac{y}{1 + y}\ , \quad dx = \frac{dy}{(1 + y)^2}\ , \quad C_1 = 1 - C \quad , \tag{D-6}$$

to get

$$A = -C_1^N \int_0^\infty dy \, \frac{(1+y)^{N-1} - y^{N-1}}{(1 + C_1 y)^N}$$

$$= -C_1^N \sum_{j=0}^{N-2} \binom{N-1}{j} \int_0^\infty dy \, \frac{y^j}{(1 + C_1 y)^N}$$

$$= -C_1^N \sum_{j=0}^{N-2} \binom{N-1}{j} \frac{j!(N-2-j)!}{C_1^{j+1}(N-1)!} = -\sum_{j=0}^{N-2} \frac{C_1^{N-1-j}}{N-1-j}$$

$$= -\sum_{k=1}^{N-1} \frac{C_1^k}{k} = -\sum_{k=1}^{N-1} \frac{1}{k}(1-C)^k \quad ; \tag{D-7}$$

we employed reference 12, eq. 3.194 3, to evaluate the integral. Thus the difficult integral in (D-2) involving powers, a logarithm, and a hypergeometric function becomes a very simple polynominal in 1-C.

If we express (D-7) for $C>0$ as

$$A = -\sum_{n=1}^{\infty} \frac{(1-C)^n}{n} + \sum_{n=N}^{\infty} \frac{(1-C)^n}{n} = \ln C + \sum_{n=N}^{\infty} \frac{(1-C)^n}{n} \quad , \tag{D-8}$$

we can upper-bound the bias (remainder) in (D-8) according to

$$\sum_{n=N}^{\infty} \frac{(1-C)^n}{n} \le \frac{(1-C)^N}{N} \left[1 + (1-C) + (1-C)^2 + (1-C)^3 + \ldots\right]$$

$$= \frac{(1-C)^N}{NC} = \frac{\exp(N \ln(1-C))}{NC} \quad \text{for } C > 0 \quad . \tag{D-9}$$

Since $C>0$, the logarithm in (D-9) is negative, leading to exponential decay of the bias with N.

Mean Square Value

To determine the variance of the logarithmic output, we need to evaluate, in addition to (D-1), the mean square value (61):

$$Q = \int_0^1 dx \, p(x) \, (\ln x)^2 \quad . \tag{D-10}$$

The expression for (D-10) is the obvious modification of (D-2) obtained by replacing $\ln x$ by $(\ln x)^2$. Then using reference 12, eq. 4.261 17, we get (using $C_1 = 1-C$)

$$Q = C_1^N \sum_{k=0}^{\infty} \frac{(N)_k \, C^k}{(1)_k} \left\{ [\psi(k+N) - \psi(k+1)]^2 + \psi'(k+1) - \psi'(k+N) \right\} . \tag{D-11}$$

Now use reference 8, eqs. 6.3.6 and 6.4.6, to find

$$Q = C_1^N \sum_{k=0}^{\infty} \frac{(N)_k\, C^k}{(1)_k} \left\{ \left(\sum_{j=0}^{N-2} \frac{1}{k+1+j} \right)^2 + \sum_{j=0}^{N-2} \frac{1}{(k+1+j)^2} \right\} . \qquad \text{(D-12)}$$

At this point, we have been unable to sum this expression in closed form for general N. However, defining

$$L = -\ln(1 - C) = \sum_{n=1}^{\infty} \frac{1}{n} C^n \quad , \qquad \text{(D-13)}$$

we find that we can evaluate the following cases in closed form for any C:

$$Q = C_1^2 \frac{L}{C} 2 \text{ for } N = 2 \ ,$$

$$Q = C_1^3 \left\{ \frac{1}{C} + \frac{3}{1-C} + \frac{L}{C}\left(1 - \frac{1}{C}\right) \right\} \text{ for } N = 3 \ ,$$

$$Q = \frac{1}{3} C_1^4 \left\{ -\frac{2}{C^2} + \frac{6}{1-C} + \frac{6}{(1-C)^2} + \frac{L}{C}\left(2 - \frac{1}{C} + \frac{2}{C^2}\right) \right\} \text{ for } N = 4 \ ,$$

$$Q = \frac{C_1^5}{12} \left\{ \frac{3}{C} + \frac{1}{C^2} + \frac{6}{C^3} + \frac{25}{1-C} + \frac{20}{(1-C)^2} + \frac{20}{(1-C)^3} + \frac{L}{C}\left(6 - \frac{2}{C} + \frac{2}{C^2} - \frac{6}{C^3}\right) \right\} \text{ for } N = 5 \ . \qquad \text{(D-14)}$$

Apparently the general solution to (D-12) is of the form

$$Q = C_1^N \left\{ \sum_{n=1}^{N-2} \left[\frac{a_n}{C^n} + \frac{b_n}{(1-C)^n} \right] + \frac{L}{C} \sum_{n=0}^{N-2} \frac{d_n}{C^n} \right\} . \qquad \text{(D-15)}$$

However we have not been able to determine the general dependence of the coefficients $\{a_n\}$, $\{b_n\}$, and $\{d_n\}$ on N; they are independent of C, of course.

The mean square value, Q, for $C = 0$ and general N follows immediately from (D-12):

$$Q = \left(\sum_{k=1}^{N-1} \frac{1}{k} \right)^2 + \sum_{k=1}^{N-1} \frac{1}{k^2} \quad \text{for } C = 0 \ . \qquad \text{(D-16)}$$

Combined with the result for average A in (D-7), there follows for the variance (65):

$$V = \sum_{k=1}^{N-1} \frac{1}{k^2} \quad \text{for } C = 0 \ . \qquad \text{(D-17)}$$

An asymptotic expansion for (D-17) for large N is derived by using Watson's lemma as follows:

$$V = \sum_{k=1}^{N-1} \int_0^\infty dt\ t\ \exp(-kt) = \int_0^\infty dt\ t\ \frac{\exp(-t) - \exp(-Nt)}{1 - \exp(-t)}$$

$$= \frac{\pi^2}{6} - \int_0^\infty dt\ e^{-Nt} \frac{t}{1 - \exp(-t)}$$

$$\sim \frac{\pi^2}{6} - \int_0^\infty dt\ e^{-Nt} \left(1 + \frac{1}{2}t\right) = \frac{\pi^2}{6} - \frac{1}{N} - \frac{1}{2N^2} \quad \text{as } N \to \infty, \quad \text{for } C = 0, \qquad \text{(D-18)}$$

where we used reference 12, eq. 3.411 9. Thus, whereas the variance of the logarithmic device output tends to zero as $N \to \infty$ if $C > 0$ (see (72)), the variance saturates at $\pi^2/6$ if $C = 0$. Thus more pieces in the MSC estimate do not lead to smaller variance for the ln x transformation when $C = 0$. To compensate for this feature, we have already observed in (58) that the average output gets arbitrarily large as N increases, when $C = 0$. This behavior is due to the singularity of the ln x transformation at $x = 0$.

Appendix E

Modified Forms For Variance

The asymptotic expansion for the variance of the MC estimate is given by (68). We rewrite it as

$$V = \frac{(1-C)^2}{2N} - \frac{(1-C)^2}{8N^2 C}\left[(1-C)^2 + 4C(1-3C)\right] + O\left(N^{-3}\right) . \tag{E-1}$$

As explained in appendix C, the singular term has a factor $(1-C)^4$ and would not overly affect the variance approximation for the larger C values if it were dropped. Thus we obtain the modified variance approximation for the MC estimate as

$$\tilde{V} \equiv \frac{(1-C)^2}{2N} - \frac{(1-C)^2(1-3C)}{2N^2} . \tag{E-2}$$

For the arc tanh $(x^{1/2})$ transformation, we rewrite (70) as

$$V = \frac{1}{2N} - \frac{(1-C)^2 - 4C}{8N^2 C} + O\left(N^{-3}\right) , \tag{E-3}$$

from which we immediately obtain, upon dropping the singular term, the modification

$$\tilde{V} \equiv \frac{1}{2N} + \frac{1}{2N^2} . \tag{E-4}$$

For the ν-th law device, we develop the bracketed term in (75) as

$$\begin{aligned}(1+C)^2 &- 6\nu(1+C)(1-C) + 6\nu^2(1-C)^2 \\ &= \left(1 - 6\nu + 6\nu^2\right)(1-C)^2 + 4C(1 - 3\nu(1-C)) .\end{aligned} \tag{E-5}$$

Substituting (E-5) in (75) and discarding the singular term (first term in bracket), we obtain the modification

$$\tilde{V} \equiv \frac{2\nu^2(1-C)^2 C^{2\nu-1}}{N} + \frac{4\nu^2(1-C)^2 C^{2\nu-1}(1-3\nu(1-C))}{N^2} . \tag{E-6}$$

Appendix F

Location of Peak of Probability Density Function

The probability density function is given by (4). If we use reference 10, page 194, eq. 8.21.3, we find that

$$p(x) \sim \left(\frac{N}{\pi}\right)^{1/2} \frac{(1 - C)^N}{2C^{1/4}} \frac{(1 - x)^{N-2}}{x^{1/4}\left(1 - \sqrt{Cx}\right)^{2N-1}} \left[1 + \frac{\left(1 - \sqrt{Cx}\right)^2}{16\left(N - \frac{3}{2}\right)\sqrt{Cx}}\right]$$

$$\text{as } N \to \infty \quad . \tag{F-1}$$

In so far as locating the peak of (F-1), we need only concern ourselves with the function

$$d_N(x) = a(x) b^N(x) \left[1 + \frac{f(x)}{N - \frac{3}{2}}\right] \quad , \tag{F-2}$$

where

$$a(x) \equiv \frac{1 - \sqrt{Cx}}{x^{1/4}(1 - x)^2} \quad , \quad b(x) \equiv \frac{1 - x}{\left(1 - \sqrt{Cx}\right)^2} \quad , \quad f(x) \equiv \frac{\left(1 - \sqrt{Cx}\right)^2}{16\sqrt{Cx}} \quad . \tag{F-3}$$

The derivative of (F-2) is (in shorthand notation)

$$d_N' = a' b^N \left[1 + \frac{f}{N - 3/2}\right] + aNb^{N-1} b' \left[1 + \frac{f}{N - 3/2}\right] + ab^N \frac{f'}{N - 3/2} \quad . \tag{F-4}$$

Setting $d_N' = 0$, we find we must solve $Q_N = 0$, where

$$Q_N \equiv N^2 ab' + N\left[a'b + ab'\left(f - \frac{3}{2}\right)\right] + \left[a'b\left(f - \frac{3}{2}\right) + abf'\right] \quad . \tag{F-5}$$

To highest order, we must set either a or b ′ to zero. But from (F-3), $a(x) \neq 0$ in (0, 1). Since

$$b'(x) = \frac{\sqrt{C/x} - 1}{\left(1 - \sqrt{Cx}\right)^3} \quad , \tag{F-6}$$

we see that to highest order, the peak of d_N occurs at

$$x_1 = C \quad . \tag{F-7}$$

Now to find the correction term to x_1 of $O(N^{-1})$, we let

$$x = x_1 + \frac{u}{N} \tag{F-8}$$

in (F-5), obtaining

$$Q_N \sim N^2\left(a_1 + a_1' \frac{u}{N}\right)\left(b_1' + b_1'' \frac{u}{N}\right) + N\left[\left(a_1' + a_1'' \frac{u}{N}\right)\left(b_1 + b_1' \frac{u}{N}\right)\right.$$

$$\left. + \left(a_1 + a_1' \frac{u}{N}\right)\left(b_1' + b_1'' \frac{u}{N}\right)\left(f_1 + f_1' \frac{u}{N} - \frac{3}{2}\right)\right] + O(1)$$

$$\sim N^2 a_1 b_1' + N\left[a_1' b_1' u + a_1 b_1'' u + a_1' b_1 + a_1 b_1'\left(f_1 - \frac{3}{2}\right)\right] + O(1)$$

$$= N\left[a_1 b_1'' u + a_1' b_1\right] + O(1) \quad , \tag{F-9}$$

and performing the last step by use of $b_1' = 0$. So, to next highest order in N, we have the correction u in (F-8) as

$$u = -\frac{a_1' b_1}{a_1 b_1''} = -\frac{a'(C)\ b(C)}{a(C)\ b''(C)} \quad . \tag{F-10}$$

We find from (F-3),

$$a(C) = \frac{1}{C^{1/4}(1 - C)} \quad , \quad a'(C) = -\frac{1 - 7C}{4C^{5/4}(1 - C)^2} \quad ,$$

$$b(C) = \frac{1}{1 - C} \quad , \quad b''(C) = \frac{-1}{2C(1 - C)^3} \quad . \tag{F-11}$$

Substitution in (F-10) yields

$$u = -\frac{1}{2}(1 - C)(1 - 7C) \quad . \tag{F-12}$$

Combining this with (F-8), we find that the location of the peak of the probability density function of the MSC estimate, $\hat{C}$, is at

$$C - \frac{(1 - C)(1 - 7C)}{2N} \quad , \tag{F-13}$$

to $O(N^{-1})$. The perturbation (F-12) has a maximum value at $C = 4/7$ of value $9/14$; thus the maximum movement of the peak is $9/(14N)$.

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